Complex Geometry Exercises

Week 2

Exercise 1. *Let (M, J) be an almost complex manifold.*

(i) Show that $\dim_{\mathbb{R}} M$ *is even and* M *is naturally oriented.*

(ii) Show that there is at most one structure of a complex manifold inducing J*.*

Exercise 2. *Show that the only compact connected complex submanifold in* $\mathbb{C}^n$ *is a point.*

Exercise 3. *Show that any Hopf surface*

$$H^2_\lambda := \mathbb{C}^2 \setminus \{(0,0)\}/(z_1, z_2) \sim \lambda(z_1, z_2), \quad \lambda \in (0,1)$$

contains many elliptic curves.

Exercise 4. *Let* $f : \mathbb{CP}^n \to \mathbb{C}^m/\Lambda$ *be a holomorphic map. Then* f *is constant.*
*(**Hint**: think about how* f *interacts with the covering maps of the tours).*

Exercise 5. *Let* (M, J) *be an almost complex manifold and consider the associated* ∂ *and* μ *operators. Prove that they satisfy the following properties:*

(i) The Leibniz rule.

(ii) ∂ *is* $\mathbb{C}$*-linear and* μ *is function linear.*

(iii) The following identities hold:

$$\mu\partial + \partial\mu = 0 \, , \qquad \partial^2 + \overline{\partial}\mu + \mu\overline{\partial} = 0 \, ,$$
$$\mu^2 = 0 \, , \qquad \mu\overline{\mu} + \overline{\partial}\partial + \partial\overline{\partial} + \overline{\mu}\mu = 0 \, .$$

(continues on the back)

Exercise 6 (Serge and Plücker embeddings)**.**

(i) *Show that there is a holomorphic embedding* $\mathbb{CP}^n \times \mathbb{CP}^m \to \mathbb{CP}^{(n+1)(m+1)-1}$

(ii) *Write down the corresponding homogeneous equations in the case* $n = m = 1$*.*

(iii) *Show that the map*

$$\begin{array}{rcl} Gr(k,n) & \to & \mathbb{P}(\Lambda^k\mathbb{C}^n) \\ W = \operatorname{span}\{w_1,\dots,w_k\} & \mapsto & [w_1 \wedge \cdots \wedge w_k] \end{array}$$

is a holomorphic embedding.

(iv) *Write down the corresponding homogeneous equations for the Grassmannian* $Gr(2,4)$*.*

Exercise 7. *Let* $B = B_r(0) \subseteq \mathbb{C}^n$ *be a polydisc. Then, the cohomology groups* $H^{p,q}_{BC}(B)$ *and* $H^{p,q}_{A}(B)$ *vanish for* $p, q > 0$*.*